Homework 1 MATH 543 Prof. Blomgren Parham Khodadi

TB-1.2: Suppose the masses m_1, m_2, m_3, m_4 are located at positions x_1, x_2, x_3, x_4 in a line and connected by springs with spring constants k_{12}, k_{23}, k_{34} whose natural lengths of extension are $\ell_{12}, \ell_{23}, \ell_{34}$. Let f_1, f_2, f_3, f_4 denote the rightward forces on the masses, *e.g.* $_1 = k_{12}((x_2 - x_1) - \ell_{12})$.

A. Write the (4×4) matrix equation relating the column vectors $\vec{f}$ and $\vec{x}$. Let K denote the matrix in this equation.
B. What are the units of the entries of K in the physics sense, *e.g.* mass× time, distance/mass, etc... There are at least two (natural) forms of the answer, you only have to provide one.
C. What are the units of $\det(K)$, again in the physics sense?
D. Suppose K is given numerical values based on the units meters, kilograms and seconds. Now the system is rewritten with a matrix K' based on centimeters, grams, and seconds. What is the relationship of K' to K? What is the relationship of $\det(K')$ to $\det(K)$?

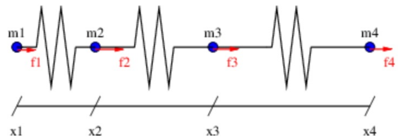

Figure: Left-to-right: Four masses connected by three springs --- $m_1, s_{12}, m_2, s_{23}, m_3, s_{34}, m_4$. Right-pulling forces f_1, f_2, f_3, f_4 are indicated on the masses. The locations x_1, x_2, x_3, x_4 of the masses are indicated.

$f_1 = k_{12}((x_2 - x_1) - \ell_{12})$

$= -k_{12}x_1 + k_{12}x_2 - k_{12}\ell_{12}$

$f_2 = -k_{12}((x_2 - x_1) - \ell_{12}) + k_{23}((x_3 - x_2) - \ell_{23})$

$= k_{12}x_1 + (-k_{12} - k_{23})x_2 + k_{23}x_3 + k_{12}\ell_{12} - k_{23}\ell_{23}$

$f_3 = -k_{23}((x_3 - x_2) - \ell_{23}) + k_{34}((x_4 - x_3) - \ell_{34})$

$= k_{23}x_2 + (-k_{23} - k_{34})x_3 + k_{34}x_4 + k_{23}\ell_{23} - k_{34}\ell_{34}$

$f_4 = -k_{34}((x_4 - x_3) - \ell_{34})$

$= k_{34}x_3 - k_{34}x_4 + k_{34}\ell_{34}$

(a) $\vec{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ $\quad K = \begin{bmatrix} -k_{12} & k_{12} & 0 & 0 \\ k_{12} & (-k_{12} - k_{23}) & k_{23} & 0 \\ 0 & k_{23} & (-k_{23} - k_{34}) & k_{34} \\ 0 & 0 & k_{34} & -k_{34} \end{bmatrix}$ $\quad c = \begin{bmatrix} -k_{12}\ell_{12} \\ k_{12}\ell_{12} - k_{23}\ell_{23} \\ k_{23}\ell_{23} - k_{34}\ell_{34} \\ k_{34}\ell_{34} \end{bmatrix}$

$\Rightarrow \boxed{\vec{f} = K\vec{x} + c}$

(b) $K = \dfrac{\vec{f} - c}{\vec{x}} = \dfrac{\text{Force}}{\text{Distance}} = \dfrac{N}{m} = \dfrac{kg \cdot m/s^2}{m} \rightarrow \boxed{[K_{ij}] = kg/s^2}$

(c) $\det(K) = ([K_{ij}])^4 = (kg/s^2)^4 = \boxed{kg^4/s^8}$

(d) $K(m, kg, s) \Rightarrow K'(cm, g, s)$ $\qquad K = kg/s \Rightarrow K' = g/s$

$1\,m = 100\,cm \qquad 1\,kg = 1000\,g$

$K' = \dfrac{1000\,g}{1\,kg} \cdot K \Rightarrow \boxed{K' = 1000\,K} \rightarrow \det(K') = (1000)^4 \det(K)$

$\Rightarrow \boxed{\det(K') = 10^{12} \det(K)}$