MATH 543 – Homework 7

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Intro

Use Gaussian Elimination with Partial Pivoting (GE w/PP) to study the growth factor (Eq. 22.2):

$$\rho = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|}$$

by creating plots analogous to TB-Figure 22.1 and TB-Figure 22.2. For computational efficiency, use the built-in `lu()` factorization with partial pivoting.

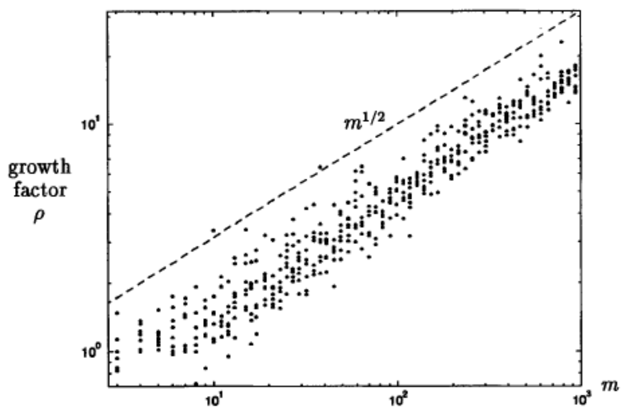

Figure 22.1. *Growth factors for Gaussian elimination with partial pivoting applied to* 496 *random matrices (independent, normally distributed entries) of various dimensions. The typical size of ρ is of order $m^{1/2}$, much less than the maximal possible value 2^{m-1}.*

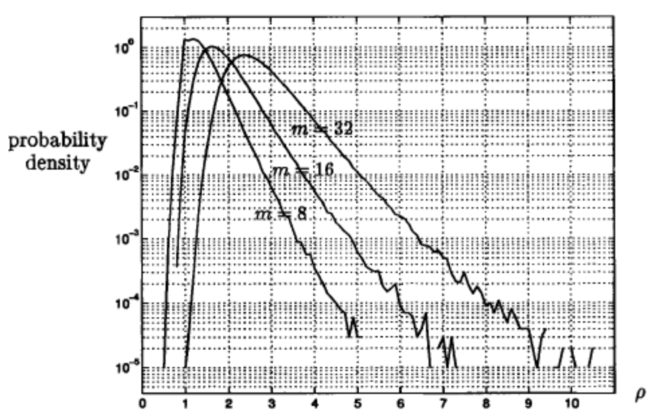

Figure 22.2. *Probability density distributions for growth factors of random matrices of dimensions m = 8, 16, 32, based on sample sizes of one million for each dimension. The density appears to decrease exponentially with ρ. The chatter near the end of each curve is an artifact of the finite sample sizes.*

6.5.1

Problem

For matrices with random, normally distributed $N(0,1)$ entries, plot the growth factor ρ for GE w/PP. (TB-Figure 22.1) — Use at least 1,024 matrices with varying sizes (up to at least 2048×2048 matrices).

Solution

MATLAB code found in Appendix.

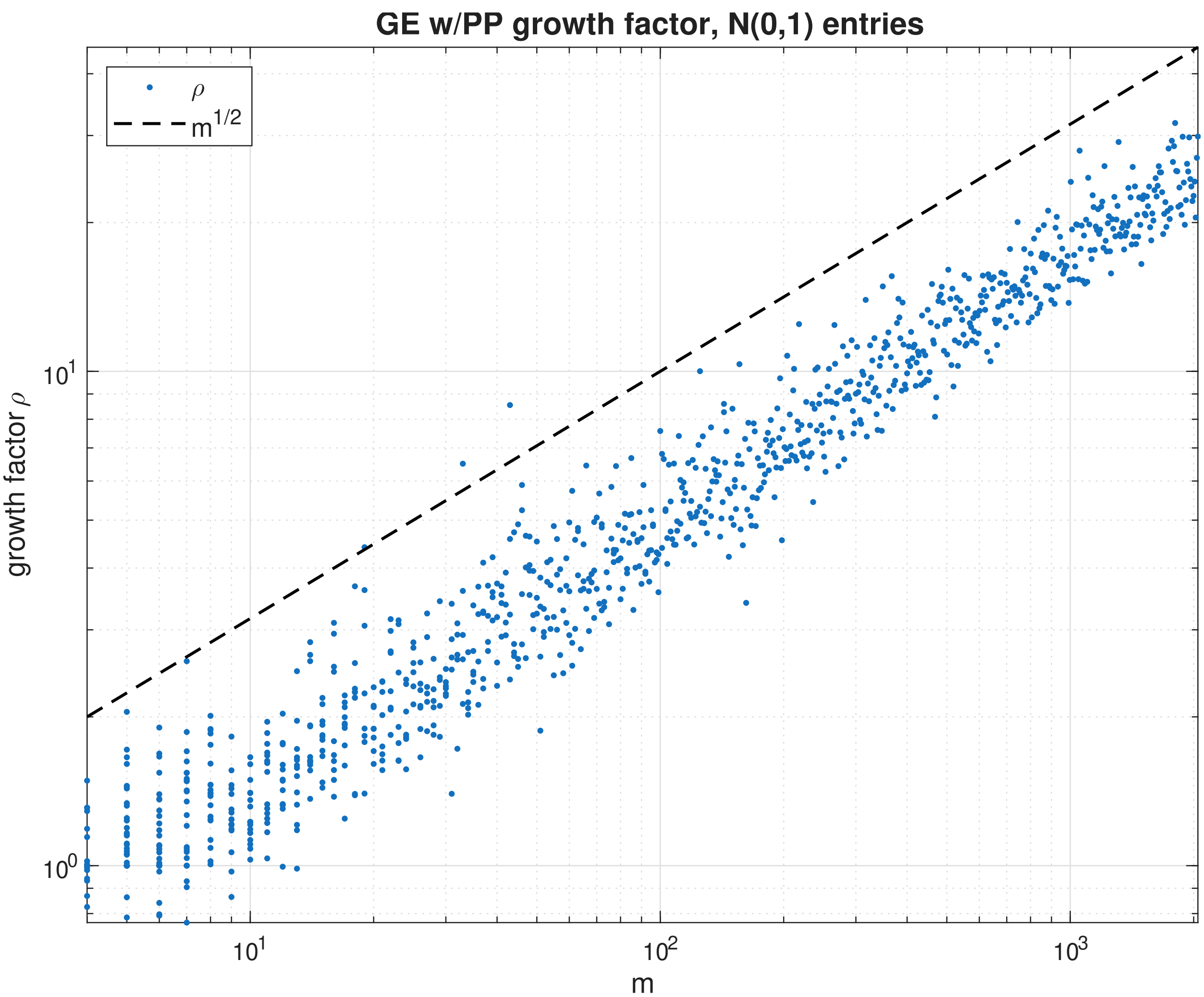

Figure 1: Growth factor ρ for GE w/PP applied to random $N(0,1)$ matrices of varying size m.

6.5.2

Problem

For matrices with random, normally distributed $N(0,1)$ entries, plot the probability density of ρ. (TB-Figure 22.2) — Use at least 1,048,576 matrices of each $m \times m$ size, $m \in \{8, 16, 32, 64\}$.

Solution

MATLAB code found in Appendix.

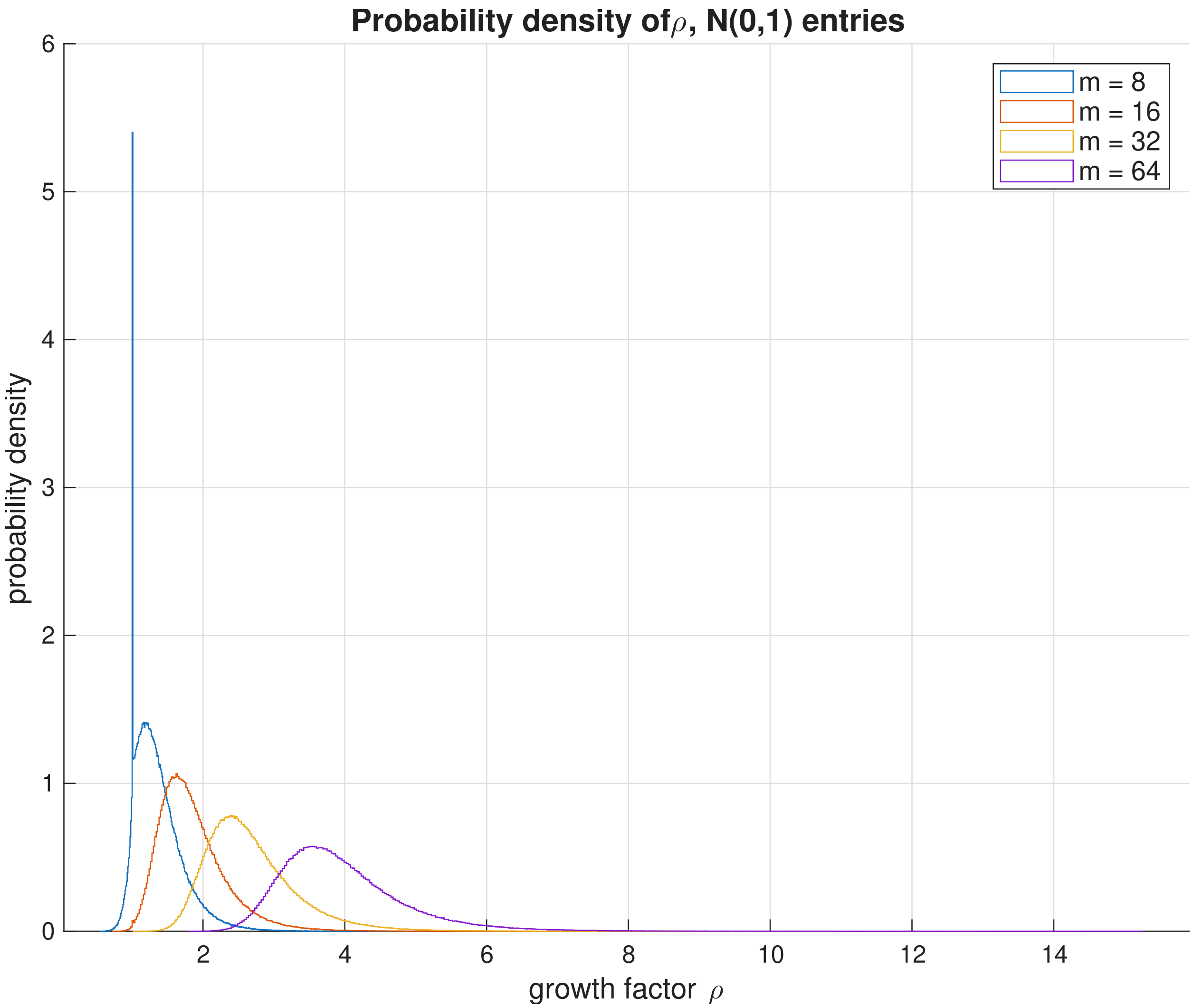

Figure 2: Probability density of growth factor ρ for $N(0,1)$ matrices with $m \in \{8, 16, 32, 64\}$.

6.5.3

Problem

For matrices with random, uniformly distributed entries in $[0, 1]$, plot the growth factor ρ for GE w/PP as a function of matrix size m (variant of TB-Figure 22.1). Use at least 1,024 matrices with varying sizes (up to at least 2048×2048 matrices).

Solution

MATLAB code found in Appendix.

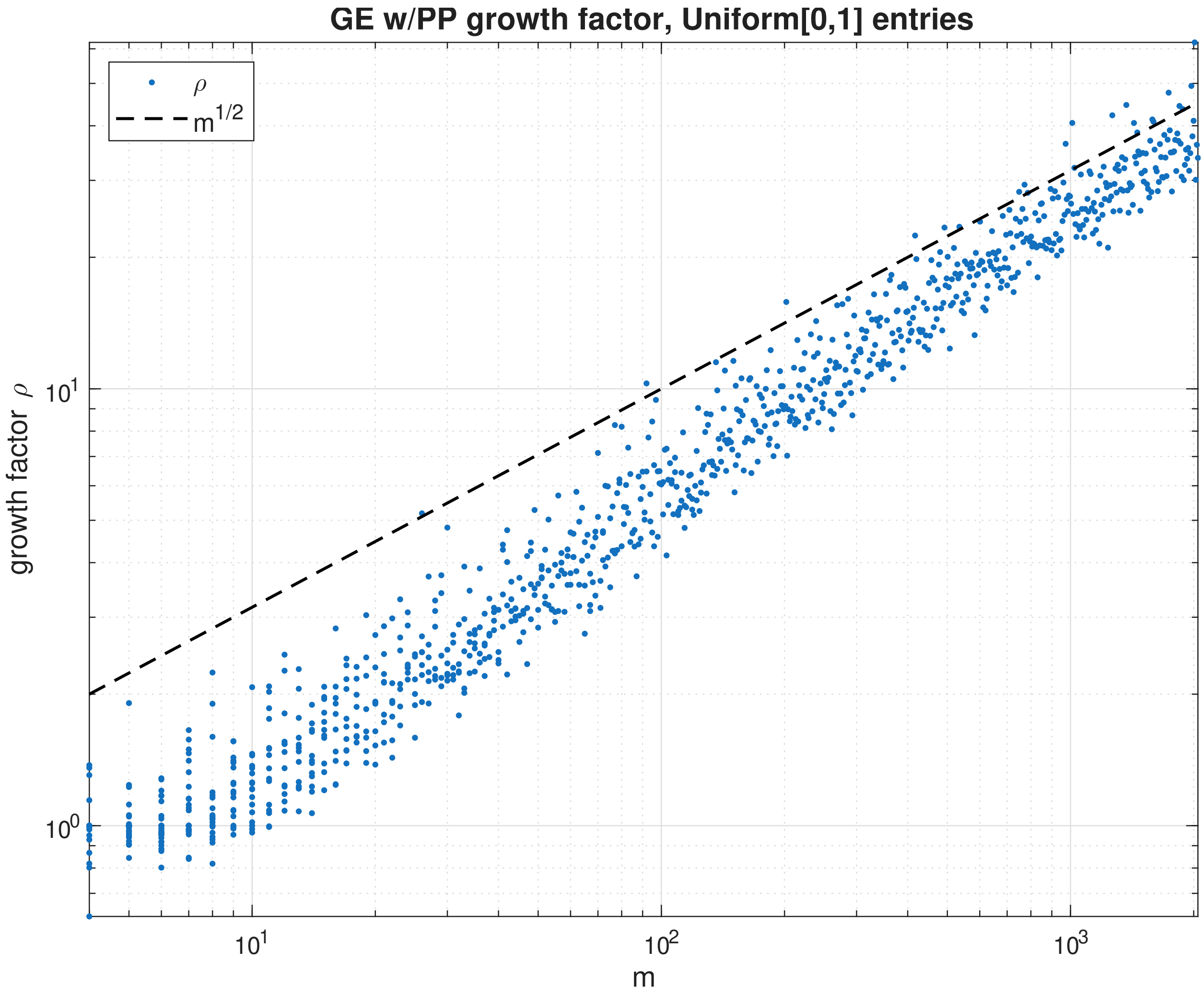

Figure 3: Growth factor ρ for GE w/PP applied to random uniform $[0, 1]$ matrices of varying size m.

6.5.4

Problem

For matrices with random, uniformly distributed entries in $[0, 1]$, plot the probability density of the growth factor ρ (variant of TB-Figure 22.2). Use at least 1,048,576 matrices of each $m \times m$ size, $m \in \{8, 16, 32, 64\}$.

Solution

MATLAB code found in Appendix.

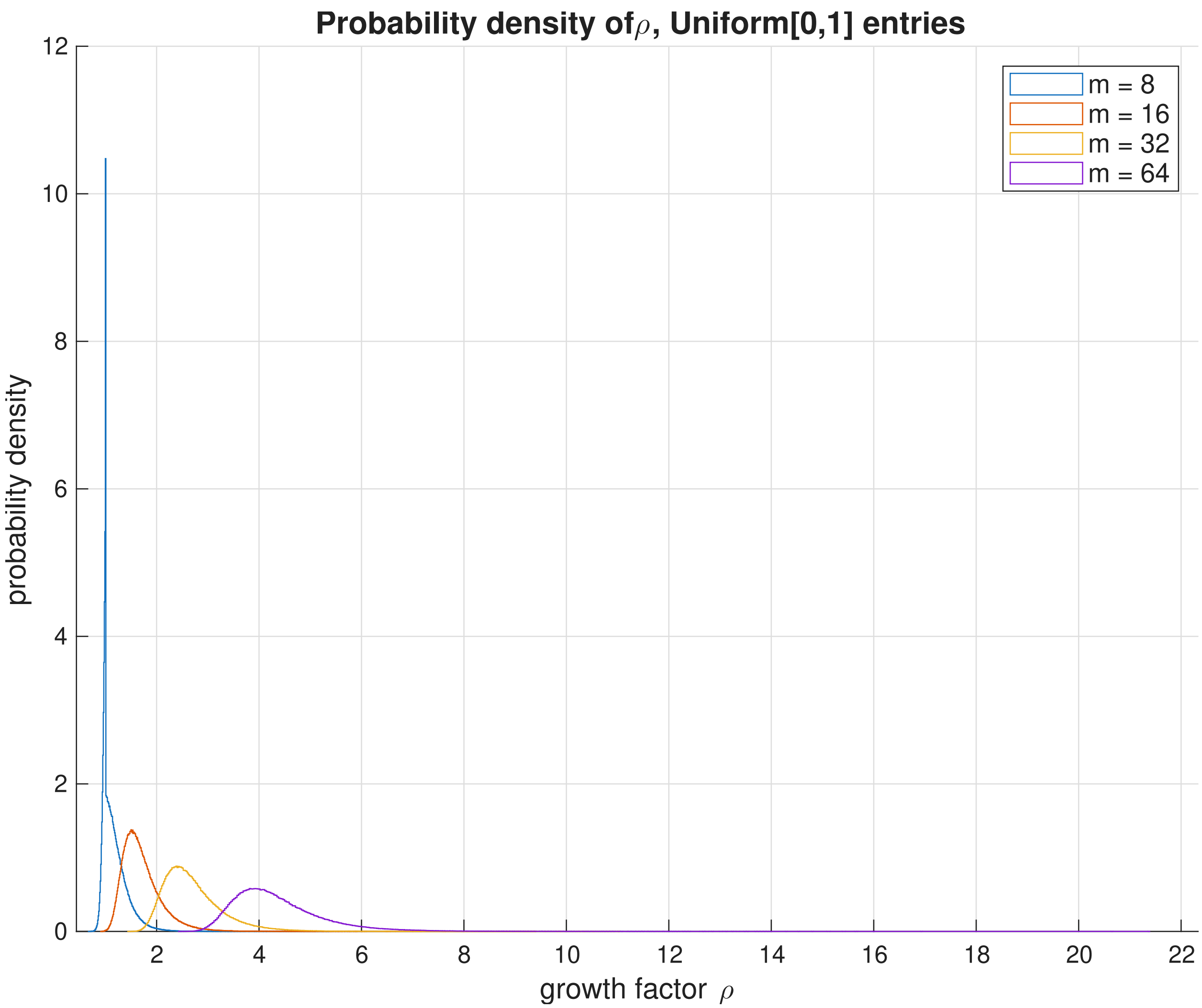

Figure 4: Probability density of growth factor ρ for uniform $[0, 1]$ matrices with $m \in \{8, 16, 32, 64\}$.

6.5.5

Problem

Comment on similarities and differences of normally distributed $N(0,1)$ vs. uniformly distributed $[0,1]$ matrix entries.

Discussion

Similarities

Figures 1 and 3 are visually almost the same plot. In both, the cloud of ρ values rides along and just below the dashed $\sqrt{m}$ line across the full range $m \in [4, 2048]$, and in both the largest observed ρ at $m = 2048$ is only around 30–40, which is nowhere near the worst-case 2^{m-1} from Lecture 17.

Figures 2 and 4 also share the same shape. For each m the density has a peak near small ρ and a right tail that decays fast, and as m grows the peak shifts right and the distribution broadens. Both entry distributions confirm that $\rho \approx \sqrt{m}$ is practically stable.

Differences

Comparing Figures 2 and 4 directly, the uniform case produces slightly more spread. The right tails in Figure 4 extend out to roughly $\rho \approx 22$ for $m = 64$, while Figure 2's tails stop near $\rho \approx 14$. The $m = 64$ mode sits near $\rho \approx 4$ for uniform entries but near $\rho \approx 3.5$ for Gaussian entries. That is a small but visible rightward shift.

Appendix: MATLAB Code

```matlab
%% MATH 543 HW 7
% By Parham Khodadi
clear; clc; close all;

%% 6.5.1
fprintf('6.5.1\n');
t0 = tic;

nTrials1     = 1024; % min size for speed
mList        = round(logspace(log10(4), log10(2048), nTrials1));
rho1         = zeros(nTrials1,1);     % pre alloc

for k = 1:nTrials1
    m        = mList(k);
    A        = randn(m); % build mxm matrix with N(0,1) entries
    U        = triu(lu(A));  % Extract U from PA=LU
    rho1(k) = max(abs(U(:))) / max(abs(A(:)));  % eq. 22.2
end
fprintf('   done in %.1f s\n', toc(t0));

f1 = figure('Position',[100 100 800 600],'Color','w');
loglog(mList, rho1, '.', 'MarkerSize', 6);
hold on;
mref = logspace(log10(4), log10(2048), 200);
loglog(mref, sqrt(mref), 'k--', 'LineWidth', 1.25); % rho = sqrt(m)

xlabel('m','FontSize',12);
ylabel('growth factor \rho','FontSize',12);
title('GE w/PP growth factor, N(0,1) entries','FontSize',13);
legend({'\rho','m^{1/2}'},'Location','northwest','FontSize',11);
grid on;
axis tight;
print(f1,'Figures/P6_5_1.eps','-depsc2');

%% 6.5.2
fprintf('6.5.2\n');
t0 = tic;

mSet         = [8 16 32 64];
nTrials2     = 1048576;

f2 = figure('Position',[100 100 800 600],'Color','w');
hold on;

for j = 1:numel(mSet)
    m     = mSet(j);
    rhos = zeros(nTrials2,1);
    for k = 1:nTrials2
        A        = randn(m); % build mxm matrix with N(0,1) entries
        U        = triu(lu(A));  % Extract U from PA=LU
        rhos(k) = max(abs(U(:))) / max(abs(A(:)));  % eq. 22.2
```

```
    end
    histogram(rhos, 'Normalization','pdf', 'DisplayStyle','stairs', 'DisplayName', sprintf('m = %d', m));
    fprintf('   m = %3d done (%.1f s elapsed)\n', m, toc(t0));
end

xlabel('growth factor  \rho','FontSize',12);
ylabel('probability density','FontSize',12);
title('Probability density of \rho, N(0,1) entries','FontSize',13);
legend('show','FontSize',11); grid on;
print(f2,'Figures/P6_5_2.eps','-depsc2');

%% 6.5.3
fprintf('6.5.3\n');
t0 = tic;

rho3 = zeros(nTrials1,1);
for k = 1:nTrials1
    m         = mList(k);
    A         = rand(m);    % diff from 6.5.1 here uses rand(m) instead of randn(m)
    U         = triu(lu(A));
    rho3(k)   = max(abs(U(:))) / max(abs(A(:)));    % eq. 22.2
end
fprintf('   done in %.1f s\n', toc(t0));

f3 = figure('Position',[100 100 800 600],'Color','w');
loglog(mList, rho3, '.', 'MarkerSize', 6);
hold on;

loglog(mref, sqrt(mref), 'k--', 'LineWidth', 1.25);
xlabel('m','FontSize',12);
ylabel('growth factor  \rho','FontSize',12);
title('GE w/PP growth factor, Uniform[0,1] entries','FontSize',13);
legend({'\rho','m^{1/2}'},'Location','northwest','FontSize',11);
grid on;
axis tight;
print(f3,'Figures/P6_5_3.eps','-depsc2');

%% 6.5.4
fprintf('6.5.4\n');
t0 = tic;

f4 = figure('Position',[100 100 800 600],'Color','w');
hold on;

for j = 1:numel(mSet)
    m    = mSet(j);
    rhos = zeros(nTrials2,1);
    for k = 1:nTrials2
        A       = rand(m);  % Same as 6.5.2 but uses rand(m) instead of randn(m)
        U       = triu(lu(A));
        rhos(k) = max(abs(U(:))) / max(abs(A(:)));  % eq. 22.2
```

```matlab
    end
    histogram(rhos, 'Normalization','pdf', 'DisplayStyle','stairs', 'DisplayName', sprintf('m = %d', m));
    fprintf('   m = %3d done (%.1f s elapsed)\n', m, toc(t0));
end

xlabel('growth factor  \rho','FontSize',12);
ylabel('probability density','FontSize',12);
title('Probability density of \rho, Uniform[0,1] entries','FontSize',13);
legend('show','FontSize',11);
grid on;
print(f4,'Figures/P6_5_4.eps','-depsc2');

fprintf('\nAll figures in ./Figures/\n');
```